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Ergodic Properties of Affine Transformations

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Let G be a locally compact abelian group. In [9] M. Rajagopalan showed that the group G must be compact if there exists on G a group automorphism which is ergodic with respect to the Haar measure. The purpose of this note is to extend this result to show that the group G must be compact if there exists on G an affine transformation which is totally ergodic with respect to some Borel measure m having the property that $m(U) > 0$ for every non-empty open subset U of G. We note that the condition of total ergodicity cannot be replaced by that of ergodicity in this assertion, in general, since a simple example of a translation map S defined on the discrete group Z of integers by $S(n) = n + 1$ gives an ergodic affine transformation on a locally compact, noncompact abelian group. We shall show, however, that if the group G is connected, then the existence of an ergodic affine transformation already implies that G must be compact.

We follow [3] and [4] for notions in ergodic theory and measure theory, and [8] for notions in topological groups. In this note we always mean by a group automorphism a *continuous* algebraic isomorphism of the group G onto G. We say that a group automorphism A is *bicontinuous* if the inverse A^{-1} is also continuous. It is known that a group automorphism of a locally compact abelian group need not be continuous [9]. A transformation T from G to G is called (Borel-) measurable if $T^{-1}(F)$ is a Borel set whenever F is. If m is a Borel measure on G, a measurable transformation T is said to be *ergodic with respect to m* if either $m(F) = 0$ or $m(G \sim F) = 0$ hold for any Borel subset F of G satisfying $T^{-1}(F) = F$. T is said to be *totally ergodic with respect to m* if for every integer $n \neq 0$ the transformation T^n is ergodic with respect to m. In this note we shall simply call a transformation T *ergodic* or *totally ergodic* if T is ergodic or totally ergodic with respect to *some* Borel measure m

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having the property that $m(U) > 0$ for every nonempty open subset U of G. It is clear that a group automorphism and a translation map defined by an element of the group are both measurable transformations. It is also known that a group automorphism which is not bicontinuous is not ergodic with respect to the Haar measure [9]. By an *affine transformation* on a locally compact abelian group G, we mean a transformation T of the form $T(x) = a + A(x)$, where a is an element of G and A is a group automorphism of G. For any subset E of G, we denote by $[E]$ the smallest *closed* subgroup of G containing E. If a is an element of G, we simply write $[a]$ for $[\{a\}]$.

LEMMA 1. *Let G be a locally compact abelian group. If $T(x) = a + A(x)$ is an affine transformation on G which is ergodic, then, A, and therefore T, must be bicontinuous.*

Proof. Since the assertion is obvious if G is discrete, we assume that G is not discrete. Let V be a compact neighborhood of the identity of G and F be the group generated by the compact set $V \cup T(V)$. Since A is continuous, the sets $A^j(F), j = 0, 1, 2,...$ are σ–compact. The subgroup H of G generated by $\bigcup_{j=0}^{\infty} A^j(F)$ is open, σ-compact, and satisfies $A(H) \subset H$. If A is not bicontinuous, we infer from [9] that $A^{-1}(H)$ is not σ-compact. Since $a \in T(V) \subset H$, we see that $T^{-1}(H) = A^{-1}(-a) + A^{-1}(H) = A^{-1}(H)$. Hence, if we write $P = T^{-1}(H) \sim H$, then P is a nonempty open and closed subset of G and is wandering under T (i.e., $T^k(H) \cap T^j(H) = \varnothing$, whenever $k \neq j$). Furthermore, if m is a Borel measure with respect to which T is ergodic, then since G is not discrete, P contains a compact subset C such that $m(C) > 0$ and $m(P \sim C) > 0$. However, this contradicts the ergodicity of T, since the set $W = \bigcup_{k=-\infty}^{\infty} T^k(C)$ is a Borel subset of G satisfying $T^{-1}(W) = W$, $m(W) > 0$ and $m(G \sim W) \geqslant m(P \sim C) > 0$.

We now state our main result.

THEOREM. *Let G be a locally compact abelian group. If there exists on G an affine transformation $T(x) = a + A(x)$ which is totally ergodic, then G must be compact.*

Proof of this theorem will be carried out in three steps.

Step I. We first consider the case when G is discrete. In this case, we show that not only is G compact but, in fact, G must consist of the identity element e alone. To prove this, we first note that the ergodicity of T implies that the set $\{T^n(e) \mid n = 0, \pm 1, \pm 2,...\}$ must coincide with G. Since T^2 is also ergodic, we also have $\{T^{2n}(e) \mid n = 0, \pm 1, \pm 2,...\} = G$. Therefore, there should be an integer n such that $T(e) = T^{2n}(e)$, which implies $T^{2n-1}(e) = e$ with $2n - 1 \neq 0$. The ergodicity of T^{2n-1} now implies that $G = \{e\}$.

Step II. We now consider the case when G is totally disconnected but nondiscrete. In this case, there exists an open compact subgroup H. Define $H_0 = H$, $H_n = \sum_{j=-n}^{n} A^j(H)$, $n = 1, 2, 3, \ldots$, then since, by Lemma 1, A is bicontinuous, $\{H_n \mid n = 0, 1, 2, \ldots\}$ forms an increasing sequence of open compact subgroups of G. If we define $H_\infty = \bigcup_{n=0}^{\infty} H_n$, then H_∞ is an open subgroup and satisfies $A(H_\infty) = H_\infty$. Since H_∞ is open, the quotient group $G' = G/H_\infty$ is discrete. Let us denote by π the canonical map $G \to G'$ and by T' the transformation induced on G' by the equation $T'(\pi(x)) = \pi(T(x))$. Since H_∞ is invariant under A, we see that T' defines an affine transformation on G'. Furthermore, T' is totally ergodic since if m is a Borel measure with respect to which T is totally ergodic, then T' is totally ergodic with respect to the induced measure $m' = m \circ \pi^{-1}$. Since G' is discrete, we now conclude, as in Step I, that $G' = \{e'\}$ where e' is the identity element of G'. This means that $G = H_\infty$. Consequently, there exists a nonnegative integer k such that $a \in H_k$. Since H_k is a compact subgroup, we conclude that the closed subgroup $[a]$ is also compact. If we now define $K = H + [a]$, then K will be an open compact subgroup of G and $a \in K$. If we write $G_n = \sum_{j=1}^{n} A^{-j}(K)$, $n = 1, 2, \ldots$, then $\{G_n \mid n = 1, 2, \ldots\}$ forms an increasing sequence of open compact subgroups of G and if we define $G_\infty = \bigcup_{n=1}^{\infty} G_n$, then G_∞ is an open subgroup satisfying $A^{-1}(G_\infty) \subset G_\infty$. Since $a \in K$, we note that $A^{-1}(-a) \in A^{-1}(K) \subset G_\infty$. Therefore,

$$T^{-1}(G_\infty) = A^{-1}(-a) + A^{-1}(G_\infty) \subset G_\infty .$$

Since the open subgroup G_∞ is closed also, we see that the set

$$P = G_\infty \sim T^{-1}(G_\infty)$$

is an open and closed subset of G wandering under T. Since G is nondiscrete, we conclude, by using the ergodicity of T, that $P = \varnothing$, and therefore, $G_\infty = T^{-1}(G_\infty)$. It follows from the ergodicity again that we must have $G_\infty = G$. Since K is compact, we can find a nonnegative integer n such that $K \subset G_n$. It then follows that $A(G_n) \subset K + G_n = G_n$. Since $a \in K$, we get $T(G_n) = a + A(G_n) \subset G_n$. By using the same argument as before, we conclude that $G_n = G$, and hence that G is compact.

Step III. We finally consider the case of an arbitrary locally compact abelian group G. Let G_0 be the component of the identity in G. Since A is bicontinuous, we have $A(G_0) = G_0$. The quotient group G/G_0 is totally disconnected. Since the total ergodicity of T implies as before that the induced affine transformation T' on G/G_0 is also totally ergodic, we conclude, by using Step I or Step II above, that the quotient group G/G_0 must be compact. However, if this is the case, it is known that such a locally compact group can be written as a direct sum $K \oplus H$, where K is the maximal compact subgroup of G and H is a closed subgroup isomorphic to a vector group R^n of some dimension n. Since K is maximal and since A is bicontinuous, we have $A(K) = K$. Therefore, T induces an affine transformation T'' on the quotient group G/K, and T'' will be totally ergodic. Since the group G/K is isomorphic to the group H, which in turn is isomorphic to R^n, in order to complete the proof of Step III it suffices to show that if $n > 0$ there can exist no affine transformation on R^n which is totally ergodic. We shall, in fact, prove, by induction on the dimension n, that there can be no ergodic affine transformation on R^n.

On R^1 any affine transformation T_1 is of the form

$$T_1(x) = a + A_1(x) = a + bx,$$

where a, $b \in R^1$. If $b = 1$, then $T_1(x) = a + x$, which cannot be ergodic since it has an open wandering set (for example, the open interval $(0, \frac{1}{2} a)$ will be wandering under T_1 if $a > 0$). If, on the other hand, $b \neq 1$, then we we can find an element $c \in R^1$ such that $(1 - b)\, c = a$. But then for any $x \in R^1$, we get $T_1(c + x) = a + b(c + x) = c + bx$, which means that the transformations T_1 and A_1 are related by the identity $T_1 \circ S_c = S_c \circ A_1$, where S_c is the translation map $S_c(x) = c + x$. From this it follows, in particular, that the transformation T_1 possesses an open wandering set if and only if A_1 does. Since it is easy to see that if $b \neq 1$, $A_1(x) = bx$ does have an open wandering set (for example, if $b > 1$, then the open interval (b, b^2) is wandering under A_1), we conclude that T_1 cannot be ergodic. This establishes the assertion for the case $n = 1$. Now, suppose the assertion were true for R^k, $0 < k \leqslant n - 1$, and suppose that $T_1(x) = a + A_1(x)$ is an affine transformation on R^n. Consider the map $\Phi(x) = (I - A_1)(x)$ defined on R^n, where $I(x)$ denotes the identity map. Clearly, $\Phi(R^n)$ is a vector subspace of R^n. If $\Phi(R^n) = R^n$, then there exists an element $c \in R^n$ such that $(I - A_1)(c) = a$. We then obtain as before the relationship $T_1 \circ S_c = S_c \circ A_1$. Now, for the automorphism A_1 on R^n we can construct, as in p. 28 of [4], a nonconstant nonnegative continuous function $f(x)$ which satisfies

$$f(A_1(x)) = |\det A_1|^2 f(x) \qquad \text{for all } x.$$

We now consider two possibilities. If $|\det A_1| = 1$, then the function $g(x)$ defined on R^n by $g(x) = f(S_c^{-1}(x))$ is continuous and satisfies $g(T_1(x)) = g(x)$ for all x. Therefore, T_1 cannot be ergodic in this case. On the other hand, if $|\det A_1| \neq 1$, then it is easy to show that the automorphism A_1 has an open wandering set (for example, if $\alpha = |\det A_1|^2 > 1$, then the open set $\{x \mid \alpha < f(x) < \alpha^2\}$ is wandering under A_1). Thus, T_1 also has an open wandering set, and therefore, T_1 cannot be ergodic in this case either. Next, if $\Phi(R^n) = \{0\}$, then $A_1 = I$, so that T_1 reduces to a translation map

$T_1(x) = a + x$, which can be easily seen to be nonergodic. Finally, if $\Phi(R^n)$ is a proper nonzero closed subspace of R^n, then the quotient group $R^n/\Phi(R^n)$ is isomorphic to R^k for some k with $0 < k \leqslant n - 1$. Since Φ and A_1 commute, we have $A_1(\Phi(R^n)) = \Phi(R^n)$. Therefore, T_1 induces an affine transformation T_1' on the quotient group $R^n/\Phi(R^n)$ and T_1' will be ergodic if T_1 is. However, by the induction hypothesis, no ergodic affine transformation can exist on $R^n/\Phi(R^n)$. Thus we conclude that there can be no ergodic affine transformation on R^n. This completes the induction and finishes the proof of Step III. Proof of the theorem is now established in all cases.

COROLLARY 1. *Let G be a locally compact connected abelian group. If there exists an ergodic affine transformation $T(x) = a + A(x)$ on G, then G must be compact.*

Note that in view of the simple example quoted in the beginning the hypothesis of connectedness is crucial in the assertion above.

Proof of Corollary 1. It is known that a locally compact connected abelian group can represented as a direct sum $K \oplus H$, where K is the maximal compact subgroup of G and H is a closed subgroup isomorphic to a vector group R^n of some dimension n. Since the maximality of K implies $A(K) = K$, T induces an ergodic affine transformation T' on the quotient group G/K. But in Step III of the proof of the theorem above it was shown that if $n > 0$, then there can be no ergodic affine transformation on R^n. Therefore, we must have $G = K$ and hence G is compact.

COROLLARY 2. *Let G be a locally compact totally disconnected abelian group. Then an affine transformation $T(x) = a + A(x)$ on G is totally ergodic if and only if the automorphism A is ergodic with respect to the Haar measure of G.*

Proof. One can show as in Step I and II of the proof of the theorem above (but with considerably simpler arguments) that the existence of an ergodic automorphism A would imply that the group G must be compact. If A is ergodic with respect to the Haar measure then the result of F. Hahn [1] implies that the affine transformation T is strongly mixing, and therefore, is totally ergodic with respect to the Haar measure. Conversely, if T is totally ergodic, then by the theorem above, G must be compact. Thus, in order to show that the automorphism A is ergodic with respect to the Haar measure, it suffices to prove, in view of a theorem by P. Halmos [4], that for every nonconstant continuous character $f(x)$ on G there exists no nonzero integer n for which $f(A^n(x)) = f(x)$ for all x. So, suppose that $f(x)$ is a nonconstant continuous character satisfying $f(A^n(x)) = f(x)$ for all x for some nonzero integer n.

We may assume that $n > 0$. Since the group G is totally disconnected and compact, there exists a positive integer k such that $f^k(x) = 1$ for all x. Since

$$T^{nk}(x) = a + A(a) + \cdots + A^{nk-1}(a) + A^{nk}(x),$$

we have

$$\begin{aligned} f(T^{nk}(x)) &= f(A^{nk}(x)) \cdot \prod_{j=0}^{nk-1} f(A^j(a)) \\ &= f(x) \cdot \prod_{p=0}^{n-1} \prod_{j=0}^{k-1} f(A^{jn}(A^p(a))) \\ &= f(x) \cdot \prod_{p=0}^{n-1} f^k(A^p(a)) = f(x), \end{aligned}$$

which means that $f(x)$ is a nonconstant continuous function invariant under T^{nk}. This contradicts the total ergodicity of T and completes the proof.

A homeomorphism T of a locally compact Hausdorff space X onto itself is called *minimal* if no nonempty proper closed subset of X is invariant under T. T is called *totally minimal* if for every nonzero integer n, the homeomorphism T^n is minimal. Properties of minimal affine transfomations on a compact abelian group have been investigated in [2, 5 and 6]. Our proofs of Lemma 1, Theorem, and Corollary 1 above carry over with minor changes to prove the following:

COROLLARY 3. *Let G be a locally compact abelian group. If there exists on G a totally minimal affine transformation, then G must be compact. Furthermore, if G is connected, then the existence of a minimal affine transformation already implies that G is compact.*

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